INVESTIGATION IN THE NONLINEAR FORMULATION OF SOME SELF-MODELLING PROBLEMS OF THE MOTION OF AN INCOMPRESSIBLE FLUID WITH A FREE SURFACE

(ISSLEDOVANIE V NELINEINOI POSTANOVKE NEKOTORYKH AVTOMODEL'NYKH ZADACH O DVIZHENII NESZHIMAEMOI ZHIDKOSTI SO SVOBODNOI POVERKHNOST'IU)

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The solution of the problem of the motion of a fluid with a free surface presents a great difficulty, as is well-known: it is difficult to satisfy a boundary condition on a boundary which must be found in the process of solution.

The problem is somewhat simplified in the case of self-modelling motions for which the hydrodynamic quantities are functions of the ratio x/t, y/t and z/t. The problem of the entry of an infinite wedge into a half-space occupied by an incompressible fluid can serve as an example. Approximate methods of solution of this problem can be found in [1-5] and others; the problem of the entry of a wedge has been considered in nonlinear formulation in [6-8].

The problem of the entry with constant velocity v_0 of a wedge of arbitrary opening angle 2α into a half-space occupied by an ideal incompressible and weightless fluid is considered below. This problem of a domain with an unknown section of boundary is reduced to the problem of determining a function, analytic in the upper half-plane, in accordance with a nonlinear boundary condition prescribed now on a known boundary - the entire real axis.

In view of the complexity of the boundary value problem obtained, the auxiliary problem of the propagation of a continuous pressure on the free surface of an incompressible fluid is considered. We note that the problem of the penetration of the wedge can be considered as a problem of the unsteady motion of a fluid generated by the propagation of some continuous pressure along the free surface which bounds the fluid. The auxiliary problem under consideration is also reduced to a nonlinear boundary value problem, some of whose particular solutions can be found by an inverse method. In the present work one exact particular solution is found.

1. The motion of a fluid arising as a result of the penetration of a wedge is a potential one. The velocity potential $\varphi(x, y, t)$ satisfies the Laplace equation

$$\partial^2\varphi / \partial x^2 + \partial^2\varphi / \partial y^2 = 0 \qquad (1.1)$$

and the following conditions on the boundary of the domain $ABCD$ (Fig. 1) of the disturbed motion of the fluid (the boundary of the domain changes with time):

a) the condition of constant pressure on the free surface

$$\partial\varphi / \partial t + (1/2) [(\partial\varphi / \partial x)^2 + (\partial\varphi / \partial y)^2] = 0 \quad \text{on } BC \qquad (1.2)$$

b) the condition of impenetrability on the solid boundary

$$\partial\varphi / \partial n = v_0 \sin\alpha \quad \text{on } AB \qquad (1.3)$$

c) the condition of symmetry

$$\partial\varphi / \partial x = 0 \quad \text{on } AD \qquad (1.4)$$

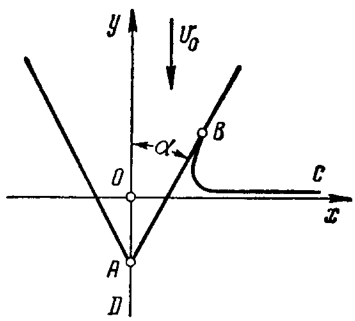

Fig. 1.

In addition, the condition

$$\partial\varphi / \partial x = \partial\varphi/\partial y = 0 \qquad \text{for } |z| \to \infty$$

and the initial condition $\varphi(x, y, 0) = 0$ must be fulfilled.

Along with condition (1.2) there is a kinematic condition on the free surface which expresses the fact that fluid particles which are found on the free surface remain on it during the entire time of the motion

$$\frac{\partial\varphi}{\partial y} - \frac{\partial f_0}{\partial x}\frac{\partial\varphi}{\partial x} - \frac{\partial f_0}{\partial t} = 0 \qquad (1.5)$$

Here $y = f_0(x, t)$ is the equation of the unknown free boundary of the fluid.

2. Let us introduce the self-modelling variables $\xi = x/v_0 t$ and $\eta = y/v_0 t$. Then

$$\varphi(x, y, t) = v_0^2 t \Phi(\xi, \eta)$$

where $\Phi(\xi, \eta)$ is a harmonic function of the variables ξ, η. The domain *ABCD* is fixed in the plane of ξ, η (Fig. 2); the equation of the free surface has the form $\eta = \eta(\xi)$.

Conditions (1.2) and (1.5) transform respectively into conditions

$$\Phi(\xi, \eta) - \xi \frac{\partial \Phi}{\partial \xi} - \eta(\xi) \frac{\partial \Phi}{\partial \eta} + \frac{1}{2}\left(\frac{\partial \Phi}{\partial \xi}\right)^2 + \frac{1}{2}\left(\frac{\partial \Phi}{\partial \eta}\right)^2 = 0 \tag{2.1}$$

$$\frac{\partial \Phi}{\partial \eta} - \eta'(\xi) \frac{\partial \Phi}{\partial \xi} + \xi \eta'(\xi) - \eta(\xi) = 0 \tag{2.2}$$

Here ξ, η are the coordinates of points on the unknown curve $\eta = \eta(\xi)$.

Let us introduce the complex potential

$$V(\zeta) = \Phi(\xi, \eta) + i\Psi(\xi, \eta), \qquad \zeta = \xi + i\eta$$

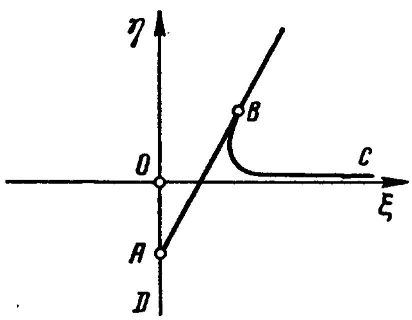

FIG. 2.

The Lagrangean integral of (2.1) can then be reduced to the form

$$\operatorname{Re}\left[V(\zeta) - \zeta V'(\zeta)\right] + \frac{1}{2} V'(\zeta) \overline{V'(\zeta)} = 0 \tag{2.3}$$

where the complex variable ζ is a point on the unknown curve $\eta = \eta(\xi)$. Conditions (1.3) and (1.4) can be written in the ζ-plane in the form

$$\operatorname{Re}\left[e^{-i\alpha} V'(\zeta)\right] = \sin \alpha \quad \text{on } AB \tag{2.4}$$

$$\operatorname{Re} V'(\zeta) = 0 \qquad \text{on } AD$$

Let us introduce a new unknown function $\zeta = \zeta(w)$ which conformally maps the upper half-plane of w (Fig. 3) onto the domain *ABCD* of the ζ-plane. The points A and B on the real axis u into which the points A and B of the ζ-plane transform can be given.

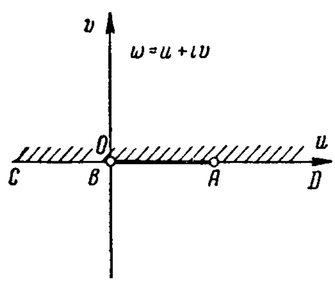

Fig. 3.

Since the parametric equations of the curve $\eta(\xi)$ have the form $\xi = \operatorname{Re} \zeta(u)$, $\eta = \operatorname{Im} \zeta(u)$, the derivative $\eta'(\xi)$ in condition (2.2) is then equal to

$$\eta'(\xi) = \frac{\operatorname{Im} \zeta'(w)}{\operatorname{Re} \zeta'(w)}\bigg|_{w=u} \tag{2.5}$$

The complex potential $V(\zeta)$ transforms into the function $V(\zeta(w))$, which we shall denote by the same letter $V(\zeta(w)) = V(w)$. In addition

$$V'(\zeta)\,|_{\zeta=\zeta(w)} = V'(w) \frac{1}{\zeta'(w)} \tag{2.6}$$

Taking (2.5) and (2.6) into consideration, we shall reduce conditions (2.2), (2.3) and (2.4) to the form

$$\frac{1}{\operatorname{Re}\zeta'(u)}\operatorname{Im}\left[\zeta'(u)\overline{\zeta(u)} - V'(u)\right] = 0 \qquad \text{on } BC\ (-\infty < u \leqslant 0,\quad v = 0) \tag{2.7}$$

$$\operatorname{Re}\left[V(u) - \frac{\zeta(u)}{\zeta'(u)}V'(u)\right] + \frac{1}{2}\frac{V'(u)\overline{V'(u)}}{\zeta'(u)\overline{\zeta'(u)}} = 0 \tag{2.8}$$

$$\operatorname{Re}\left[e^{-i\alpha}V'(u)\frac{1}{\zeta'(u)}\right] = \sin\alpha \quad \text{on } BA\ (0 \leqslant u \leqslant u_A,\ v = 0) \tag{2.9}$$

$$\operatorname{Re}\frac{V'(u)}{\zeta'(u)} = 0 \qquad \text{on } AD\ (u_A \leqslant u < +\infty,\ v = 0) \tag{2.10}$$

Conditions (2.7) to (2.10) are given on the real axis u. For mapping the function $\zeta(w)$ we have, in addition, the conditions that sections BA and AD of the u-axis transform into known parts of the boundary of the domain $ABCD$.

$$\operatorname{Re}\left[e^{+i\alpha}\zeta(u)\right] = \sin\alpha \qquad \text{on } BA, \qquad \operatorname{Re}\zeta(u) = 0 \qquad \text{on } AD \tag{2.11}$$

The argument of the function $\zeta'(w)$ is known on BA and AD, by virtue of which

$$\zeta'(w) = -ie^{-i\alpha}|\zeta'| \qquad \text{on } BA, \qquad \zeta'(w) = -i|\zeta'| \qquad \text{on } AD \tag{2.12}$$

Using (2.12), we shall transform conditions (2.7) to (2.11) to the form

$$\operatorname{Re}\left[iV'(u)\right] = \operatorname{Re}\left[i\zeta'(u)\overline{\zeta(u)}\right] \qquad \text{on } CB\ (-\infty < u \leqslant 0,\ v = 0) \tag{2.13}$$

$$\operatorname{Re}\left[V(u)\zeta'(u)\overline{\zeta'(u)} - \zeta(u)\overline{\zeta'(u)}V'(u)\right] + {}^1/_2 V'(u)\overline{V'(u)} = 0 \tag{2.14}$$

$$\operatorname{Re}\left[iV'(u)\right] = \sin\alpha\,|\zeta'(u)|, \quad \operatorname{Re}\left[e^{i\alpha}\zeta(u)\right] = \sin\alpha \qquad \text{on } BA\ (0 \leqslant u \leqslant u_A,\ v = 0)$$

$$\operatorname{Re}\left[iV'(u)\right] = 0, \qquad \operatorname{Re}\zeta(u) = 0 \qquad \text{on } AD\ (u_A \leqslant u < +\infty,\ v = 0) \tag{2.15}$$

The initial problem has been reduced to the problem of determining two functions $V(w)$ and $\zeta(w)$, which are analytic in the upper half-plane, in accordance with the boundary conditions (2.13) to (2.15) given on the real axis u of the w-plane, to a nonlinear boundary value problem of the Poincaré type for the upper half-plane.

Eliminating one of the unknown functions from conditions (2.13) to (2.15), we shall reduce the problem under consideration to a boundary value problem for a single function.

3. From conditions (2.13) to (2.15) it is obvious that the real part of the function $iV'(w)$ is expressed by the function $\zeta(w)$ on the entire real axis (the first condition on each section of the u-axis)

$$\begin{aligned} \operatorname{Re}\,[iV'(w)]_{w=u} &= \operatorname{Re}\,(i\zeta'\bar{\zeta}) && (-\infty < u \leqslant 0) \\ \operatorname{Re}\,[iV'(w)]_{w=u} &= \sin\alpha\,|\,\zeta'(u)\,| && (0 \leqslant u \leqslant u_A) \\ \operatorname{Re}\,[iV'(w)]_{w=u} &= 0 && (u_A \leqslant u < +\infty) \end{aligned} \tag{3.1}$$

Using these conditions, we shall express the complex potential in terms of the mapping function $\zeta(w)$ with the help of the Schwartz integral for the upper half-plane

$$V'(w) = -\frac{1}{\pi}\int_{-\infty}^{+\infty} \mu(u)\,\frac{du}{u-w} \tag{3.2}$$

Here $\mu(u)$ is a real, piece-wise continuous function which has discontinuities of the first kind at the points $u = 0$ and $u = u_A$

$$\mu(u) = \begin{cases} \operatorname{Re}\,(i\zeta'\bar{\zeta}) & (-\infty < u \leqslant 0) \\ \sin\alpha\,|\,\zeta'(u)\,| & (0 \leqslant u \leqslant u_A) \\ 0 & (u_A \leqslant u < +\infty) \end{cases}$$

The Schwartz integral determines the function $iV'(w)$ to within an imaginary additive. But the integral in expression (3.2) and the function $V'(w)$ vanish at infinity, by virtue of which the constant must be zero. Integrating (3.2) with respect to w (this is possible as long as the point w is found in the upper half-plane), we obtain

$$V(w) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \ln(u-w)\,\mu(u)\,du + c \tag{3.3}$$

Here c is a complex constant. Formula (3.3) expresses the complex potential $V(w)$ in the w-plane in terms of the mapping function $\zeta(w)$. In addition, if $V(w)$ and $\zeta(w)$ are related by expression (3.3), the first conditions of (2.13) to (2.15) are then fulfilled. Let us now eliminate the function $V(u)$ from the second conditions of (2.13) to (2.15), making use of expressions (3.2) and (3.3) for this purpose.

Let us find the limiting values of the functions $V(w)$ and $V'(w)$ as $w \to u_0$ (u_0 is a point on the real axis u). The value of $V(u_0)$ at the point u_0 of the real axis u is obtained by the direct substitution of u_0 for w in formula (3.3) so that

$$V(u_0) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \ln(u-u_0)\,\mu(u)\,du + c \tag{3.4}$$

For the limiting value of the function $V'(w)$ as $w \to u_0$ from the upper half-plane we have, according to the formula of Sokhotskii [9]

$$V'(u_0) = -i\mu(u_0) - \frac{1}{\pi}\int_{-\infty}^{+\infty} \mu(u)\frac{du}{u-u_0} \tag{3.5}$$

The integral in expression (3.5) is understood in the sense of the Cauchy principal value. Substituting the limiting values of the functions V and V' according to formulas (3.4) and (3.5) into the second condition of (2.13), we obtain after some re-arranging

$$\operatorname{Re}\left\{\frac{1}{\pi}\zeta'(u_0)\overline{\zeta'(u_0)}\int_{-\infty}^{+\infty}\mu(u)\ln|u-u_0|\,du + \frac{1}{\pi}\zeta(u_0)\overline{\zeta'(u_0)}\int_{-\infty}^{+\infty}\mu(u)\frac{du}{u-u_0}\right\} \tag{3.6}$$

$$+\frac{1}{2}\left[\frac{1}{\pi}\int_{-\infty}^{+\infty}\mu(u)\frac{du}{u-u_0}\right]^2 - \frac{1}{2}[\operatorname{Re} i\zeta'(u_0)\overline{\zeta(u_0)}]^2 + c_1\zeta'(u_0)\overline{\zeta'(u_0)} = 0, \qquad -\infty < u_0 \leqslant 0$$

Here c_1 is a real **constant** ($c_1 = \pi^{-1}\operatorname{Re} c$).

Besides condition (3.6) we have for the function $\zeta(w)$ on the positive part of the real axis u the second conditions of (2.14) and (2.15)

$$\operatorname{Re}[e^{i\alpha}\zeta(u_0)] = \sin\alpha \quad (0 \leqslant u_0 \leqslant u_A), \qquad \operatorname{Re}\zeta(u_0) = 0 \quad (u_A \leqslant u_0 < +\infty) \tag{3.7}$$

Condition (3.6) is a transformed expression of the Lagrange integral, and it is satisfied on the negative portion ($-\infty < u_0 \leqslant 0$) of the real axis u, which corresponds to the unknown free boundary of the fluid in the ζ-plane. Conditions (3.7) represent geometric conditions for the mapping function $\zeta(w)$.

Thus, the problem of a domain with an unknown portion of boundary has been reduced to a boundary value problem of determining a function $\zeta(w)$, analytic in the upper half-plane, in accordance with the nonlinear conditions (3.6) and (3.7) which are given on a known boundary, on the entire real axis.

4. Let us consider the problem of the propagation of a continuous pressure along the free surface of an incompressible fluid.

Let some continuous pressure begin to be propagated along a free surface at the moment $t = 0$ from a point 0 of the surface.

Let us locate the origin of the Cartesian coordinates x, y at the point 0 with the x-axis directed horizontally along the undisturbed free surface and the y-axis directed vertically upward. In order that the problem be self-modelling, we shall consider that the function $p(x, y, t)$ which gives the pressure distribution on the free surface is a function only of the ratios x/v_0t, y/v_0t (v_0 = const). The surface of the fluid is deformed under the action of the applied pressure, its form is not known and must be determined in the process of solution.

The fluid motion under consideration is a potential one. The problem of determining the disturbed fluid pressure is reduced to finding a velocity potential $\varphi(x, y, t)$, a function which is harmonic with respect to the variables x, y in the domain occupied by the fluid and which satisfies the following conditions:

the condition on the unknown deformed surface of the fluid

$$\frac{\partial\varphi}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2\right]+\frac{p\,(x, y, t)}{\rho}=0 \tag{4.1}$$

the condition

$$\frac{\partial\varphi}{\partial x}=\frac{\partial\varphi}{\partial y}=0 \qquad \text{for } |z|\to\infty$$

and an initial condition

$$\varphi\,(x, y, 0)=0$$

It is understood that p denotes throughout the excess pressure above the initial constant pressure.

On the free surface there is, along with condition (4.1), the condition of impenetrability of the fluid surface

$$\frac{\partial\varphi}{\partial y}-\frac{\partial f_0}{\partial x}\frac{\partial\varphi}{\partial x}-\frac{\partial f_0}{\partial t}=0 \tag{4.2}$$

Here $y = f_0(x, t)$ is the equation of the unknown boundary of the fluid.

5. In self-modelling variables the function $p(x, y, t)$ will have the form

$$p\,(x, y, t)\,|_S=\rho v_0^2\bar{p}\,(\xi, \eta)\,|_{\eta=\eta(\xi)}=\rho v_0^2 p\,(\xi)$$

on the surface of the fluid.

Let us introduce the complex potential $V(\zeta) = \Phi(\xi, \eta) + i\Psi(\xi, \eta)$ $(\zeta = \xi + i\eta)$ and consider the plane of the auxiliary variable w. Let the function $\zeta = \zeta(w)$ conformally map the upper half-plane of w onto the domain of the fluid motion in the ζ-plane so that the point $\zeta = \infty$ transforms into the point $w = \infty$.

From condition (4.2), in the same way as in the previous problem, let us express the functions $V'(w)$ and $V(w)$ in terms of the mapping function

$$V'(w) = -\frac{1}{\pi}\int_{-\infty}^{+\infty} \operatorname{Re}(i\zeta'\bar{\zeta})\frac{du}{u-w} \tag{5.1}$$

$$V(w) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \operatorname{Re}(i\zeta'\bar{\zeta})\ln(u-w)\,du + c \tag{5.2}$$

Here c is a complex constant.

After finding the limiting values of the functions $V(w)$ and $V'(w)$ on the real axis and eliminating them from condition (4.1) (transformed beforehand, as in the problem of the wedge), we obtain

$$\operatorname{Re}\left\{\frac{1}{\pi}\zeta'\bar{\zeta}'\int_{-\infty}^{+\infty}\operatorname{Re}(i\zeta'\bar{\zeta})\ln|u-u_0|\,du + \frac{1}{\pi}\zeta\bar{\zeta}'\int_{-\infty}^{+\infty}\operatorname{Re}(i\zeta'\bar{\zeta})\frac{du}{u-u_0}\right\} +$$
$$+\frac{1}{2}\left[\frac{1}{\pi}\int_{-\infty}^{+\infty}\operatorname{Re}(i\zeta'\bar{\zeta})\frac{du}{u-u_0}\right]^2 - \frac{1}{2}\left[\operatorname{Re}(i\zeta'\bar{\zeta})\right]^2 + \zeta'\bar{\zeta}'p(\operatorname{Re}\zeta) + c_1\zeta'\bar{\zeta}' = 0 \tag{5.3}$$

Here c_1 is a real constant.

The initial problem has been reduced to a boundary value problem for a function $\zeta(w)$ which is analytic in the upper half-plane $\operatorname{Im} w > 0$ and which satisfies condition (5.3) on the entire real axis.

6. After giving the form of the free surface in the ζ-plane and selecting an analytic function $\zeta(w)$ which maps the upper half-plane of w onto the domain of the ζ-plane bounded by the given curve, it is possible to obtain from equation (5.3) the pressure generated by the given deformation of the surface, i.e. to obtain a particular solution of the problem by an inverse method; in addition, the following circumstance should be noted.

There enters into expression (5.3) an integral of the Cauchy type and also an integral with a logarithmic kernel and the same density. The function represented by the integral of the Cauchy type vanishes at an infinitely removed point, the coefficient of w^{-1} in the expansion of this function in the neighborhood of an infinitely removed point being equal up to a constant multiplier to the integral

$$a_1 = \int_{-\infty}^{+\infty}\operatorname{Re}(i\zeta'\bar{\zeta})\,du$$

If $a_1 \neq 0$, the existence of the integral with the logarithmic kernel leads to the condition that the pressure obtained from equation (5.3) will have a logarithmic singularity at infinity, i.e. it will be infinitely large, generating however a finite displacement of the particles of the free surface. In order to avoid this singularity, the function $\zeta(w)$ must satisfy the condition

$$\int_{-\infty}^{+\infty} \operatorname{Re}(i\zeta'\bar{\zeta})\, du = 0 \tag{6.1}$$

Equation (6.1) represents the condition that the positive and negative areas bounded by the curve of the free surface and the real axis are equal. In other words, (6.1) is the consequence of the condition of incompressibility of the fluid.

7. Let the surface of the fluid be represented in the ζ-plane by the curve S, shown in Fig. 4.

Let us find the pressure whose propagation along the free surface generates a prescribed deformation of the surface of the fluid. The function $\zeta(w)$ which conformally maps the upper half-plane of w onto the region bounded by the given curve S in the ζ-plane has the form

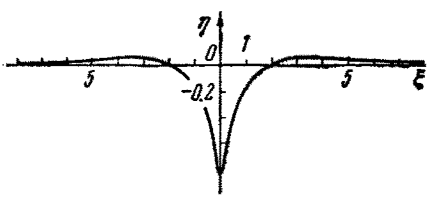

Fig. 4.

$$\zeta(w) = -w + \frac{i\alpha}{(w + i\beta)^2} + \frac{\gamma}{w + i\delta} \tag{7.1}$$

Here α, β, γ, δ are real constants, where $\beta > 0$, $\delta > 0$. These parameters must be selected so that the function $\zeta(w)$ satisfies condition (6.1) and also the condition of one-sheetedness $|\zeta'(w) \neq 0|$ everywhere in the upper half-plane

$$\left[1 + \frac{2i\alpha}{(w + i\beta)^3} + \frac{\gamma}{(w + i\delta)^2}\right] \neq 0 \qquad \text{for } \operatorname{Im} w > 0 \tag{7.2}$$

Calculating the integral (6.1) with the help of the theory of residues, we obtain from (6.1) the equation

$$\frac{3\alpha^2}{8\beta^4} + \frac{4\alpha\gamma}{(\beta + \delta)^3} + \frac{\gamma^2}{4\delta^2} - \gamma = 0 \tag{7.3}$$

which α, β, γ and δ must satisfy. The parametric equation of the curve S is

$$\xi = -u + \frac{2\alpha\beta u}{(u^2 + \beta^2)^2} + \frac{\gamma u}{u^2 + \delta^2}, \qquad \eta = \frac{\alpha(u^2 - \beta^2)}{(u^2 + \beta^2)^2} - \frac{\gamma\delta}{u^2 + \delta^2} \tag{7.4}$$

The pressure generated by the prescribed deformation of the surface is determined from equation (5.3)

$$p(u) + \frac{1}{\pi}\frac{\operatorname{Re}\zeta\overline{\zeta'}}{\zeta'\overline{\zeta'}}\int_{-\infty}^{+\infty}\operatorname{Re}(i\zeta'\bar{\zeta})\frac{du_1}{u_1-u} + \frac{1}{2}\frac{1}{\zeta'\overline{\zeta'}}\left[\frac{1}{\pi}\int_{-\infty}^{+\infty}\operatorname{Re}(i\zeta'\bar{\zeta})\frac{du_1}{u_1-u}\right]^2 +$$
$$+\frac{1}{\pi}\int_{-\infty}^{+\infty}\operatorname{Re}(i\zeta'\bar{\zeta})\ln|u_1-u|\,du_1 - \frac{1}{2}\frac{[\operatorname{Re}(i\zeta'\bar{\zeta})]^2}{\zeta'\overline{\zeta'}} + c_1 = 0 \qquad (7.5)$$

The real constant c_1 is found from the condition

$$p = 0 \qquad \text{for } |u| \to \infty \qquad (7.6)$$

and the singular integral with the Cauchy kernel and the integral with the logarithmic kernel are computed with the help of the theory of residues and have the form

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}\operatorname{Re}(i\zeta'\bar{\zeta})\frac{du_1}{u_1-u} = 2\alpha\left[\frac{3\alpha}{8\beta^4}+\frac{2\gamma}{(\beta+\delta)^3}\right]\frac{u}{u^2+\beta^2} - 2\gamma\left[1-\frac{2\alpha}{(\beta+\delta)^3}-\frac{\gamma}{4\delta^2}\right]\frac{u}{u^2+\delta^2} -$$
$$-4\alpha\beta\left[1-\frac{\alpha}{4\beta^3}-\frac{\gamma}{(\beta+\delta)^2}\right]\frac{u}{(u^2+\beta^2)^2} + \gamma(\gamma+2\delta^2)\frac{u}{(u^2+\delta^2)^2} + \frac{4\alpha\beta u(\beta^2-u^2)+2\alpha^2 u}{(u^2+\beta^2)^3} +$$
$$+\frac{2\alpha\gamma u\,[\beta(u^2-\delta^2)-\delta(u^2-\beta^2)]}{(u^2+\delta^2)^2(u^2+\beta^2)^2} - \frac{2\alpha\gamma\delta u(u^2-3\beta^2)+2\alpha\beta\gamma u(\beta^2-3u^2)}{(u^2+\beta^2)^3(u^2+\delta^2)}$$

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}\operatorname{Re}(i\zeta'\bar{\zeta})\ln|u_1-u|\,du_1 = \alpha\left[\frac{3\alpha}{8\beta^4}+\frac{2\gamma}{(\beta+\delta)^3}\right]\ln\frac{u^2+\beta^2}{u^2+\delta^2} +$$
$$+2^{\alpha}\beta\left[2-\frac{\alpha}{4\beta^3}-\frac{\gamma}{(\beta+\delta)^2}\right]\frac{1}{u^2+\beta^2} - \frac{\gamma(\gamma+2\delta^2)}{2}\frac{1}{u^2+\delta^2} - \frac{\alpha(4\beta^3+\alpha)}{2}\frac{1}{(u^2+\beta^2)^2} +$$
$$+\frac{2\alpha\gamma\beta\delta(\delta-\beta)}{(\beta^2-\delta^2)^2}\left[\frac{1}{u^2+\beta^2}+\frac{1}{u^2+\delta^2}\right] + \frac{\alpha\gamma\delta}{(\beta^2-\delta^2)^2}\frac{u^2+\beta^2}{u^2+\delta^2} -$$
$$-\frac{\alpha\gamma(7\beta^2\delta+\delta^3-4\beta^3-4\beta\delta^2)}{(\beta^2-\delta^2)^3}\frac{u^2+\delta^2}{u^2+\beta^2} + \frac{2\alpha\gamma\beta^2(\delta-\beta)}{(\beta^2-\delta^2)^3}\left(\frac{u^2+\delta^2}{u^2+\beta^2}\right)^2$$

The following combination of parameters, for example, satisfies equation (7.3):

$$\alpha = 3.2003, \qquad \beta = 2, \qquad \gamma = 0.25, \qquad \delta = 5 \qquad (7.7)$$

Moreover, the inequality (7.2) is also fulfilled because the modulus of the function representing the sum of the second and third terms in the expression (7.2) cannot be greater than 0.801 everywhere in the upper half-plane Im $w > 0$ for the prescribed parameters α, β, γ and δ.

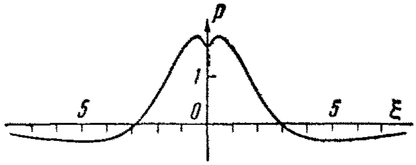

Fig. 5.

For the prescribed parameters α, β,

γ and δ we obtain from condition (7.6) that $c_1 = -0.00829$. The curve of the pressure in dimensionless form has been computed for the parameters of (7.7) according to formula (7.5) and is presented in Fig. 5.

Thus, if the pressure represented by the curve shown in Fig. 5 is applied to the free surface of a fluid, an exact solution to the problem of the generation of the fluid motion is given by formulas (5.2), (7.1) and (7.4).

Formulas (5.2) and (7.1) determine the complex velocity potential in the entire domain of the fluid motion (in the w-plane), and formulas (7.4) give the parametric equation of the curve of the deformed surface of the fluid in the ζ-plane (Fig. 4). The function $\zeta(w)$ given by formula (7.1) maps the upper half-plane $\operatorname{Im} w > 0$ onto a domain near the domain occupied by the fluid during the entry of a wedge into it. This circumstance gives the possibility of using the obtained function (7.1) as the zeroth approximation for finding a solution to the nonlinear integro-differential equation in the case of the problem of the penetration of a wedge.

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